"A Generalisation of the Functions $\Gamma(n)$ and x^n." By Rev. F. H. Jackson, R.N. Communicated by Professor J. Larmor, Sec. R.S. Received December 7,—Read December 10, 1903. Received in revised form June 15, 1904.

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It is interesting to develope from simple principles a generalisation of the functions x^n and $\Gamma(n)$. Consider an infinite sequence $(1, p, p^2, p^3, \ldots p^n, \ldots)$, then write

$$
\begin{aligned}
[1] &= 1, \\
[2] &= 1+p, \\
\ldots &\quad \ldots \\
[n] &= 1+p+p^2+\ldots+p^{n-1} \text{ (n positive and integral),} \\
[-n] &= -p^{-1}-p^{-2}-\ \ldots\ -p^{-n} \text{ (n integral).}
\end{aligned}
$$

In general for all values of x, we take $[x] = (p^x-1)/(p-1)$. The object of this note is to carry on this extension, to determine the generalised forms of the gamma function, and to investigate some of its properties.

1. Consider the expression

$$[1][2][3]\ldots[n] = [n]!.$$

We can form a function $[n]!$ in general, which is

$$\Gamma_p([n+1]) = \underset{\kappa=\infty}{\mathrm{L}} \frac{[1][2][3]\ldots[\kappa]}{[n+1][n+2][n+3]\ldots[n+\kappa]}[\kappa]^n p^{\frac{1}{2}n(n+1)} \quad (p>1) \qquad \ldots\ldots\ldots\ldots\ldots \quad (1).$$

The infinite product is convergent if $p>1$. Wherever in this expression terms of the type p^x occur, the principal value of p^x alone is meant. If n be a negative integer, the product is clearly divergent.

The Difference Equation.—We have

$$\Gamma_p([n+1]) = [n]\Gamma_p([n]) \times \underset{\kappa=\infty}{\mathrm{L}} \frac{[\kappa]}{[n+\kappa]} p^n.$$

But $\underset{\kappa=\infty}{\mathrm{L}} \frac{[\kappa]}{[n+\kappa]} p^n = \underset{\kappa=\infty}{\mathrm{L}} \frac{p^\kappa-1}{p^{n+\kappa}-1} p^n = 1 \ (p>1) = p^n \ (p<1).$

Therefore

$$\Gamma_p([n+1]) = [n \;\; \Gamma_p([n]) \quad (p>1).$$

Since the infinite product (1) is also convergent, if $p<1$, the expression for the function in this case is

$$\Gamma_p([n+1]) = \underset{\kappa=\infty}{\mathrm{L}} \frac{[1][2][3]\ldots[\kappa]}{[n+1][n+2]\ldots[n+\kappa]}[\kappa]^n \quad (p<1) \quad \ldots (2).$$

In the limit when $p = 1$, these expressions (1) and (2) reduce to Gauss's expression for Euler's gamma function.

2. *The Weierstrassian Form for the Function*—We obtain without difficulty from (1) that

$$\Gamma_p([x]) = p^{\frac{1}{2}x(x-1)} \frac{1}{[x]} \underset{\kappa=\infty}{\mathrm{L}} \left\{ e^{x\left(\log[\kappa+1]-\kappa\log p-1-\frac{1}{[2]}-\frac{1}{[3]}-\ldots-\frac{1}{[\kappa]}\right)} \right\}$$

$$\prod_{s=1}^{s=\kappa} \left\{ \left(1+p^{-x}\frac{[x]}{[s]}\right)^{-1} e^{\frac{x}{[s]}} \right\} \quad \ldots\ldots\ldots \quad (3).$$

Since $p>1$,

$$\underset{\kappa=\infty}{\mathrm{Limit}} \left\{ \log \frac{[\kappa+1]}{p^{\kappa}} \right\} = \log_e \frac{p}{p-1},$$

also the series $1+\frac{1}{[2]}+\frac{1}{[3]}+\ldots$ is absolutely convergent, so that we finally write

$$\frac{1}{\Gamma_p([x])} = p^{\frac{1}{2}x(1-x)}[x]\, e^{\mathrm{P}x} \prod_{s=1}^{=\infty} \left\{ \left(1+p^{-x}\frac{[x]}{[s]}\right) e^{-\frac{x}{[s]}} \right\},$$

in which

$$\mathrm{P} = 1+\frac{1}{[2]}+\frac{1}{[3]}+\ldots \textit{ad inf.} - \log_e \frac{p}{p-1}.$$

In the case when $p<1$

$$\frac{1}{\Gamma_p([x])} = [x]\, e^{\mathrm{Q}x} \prod_{s=1}^{s=\infty} \left\{ \left(1+p^s\frac{[x]}{[s]}\right) e^{-\frac{p^s}{[\,]}x} \right\} \quad \ldots\ldots\ldots \quad (4).$$

$$\mathrm{Q} = p+\frac{p^2}{[2]}+\frac{p^3}{[3]}+\ldots \textit{ad inf.} - \log_e \frac{1}{1-p}.$$

P and Q are extended forms of Euler's constant γ.

3. *Multiplication Theorem.*—Since $\Gamma_{p^{\frac{1}{n}}}([nx+n])$ may be written $\frac{q^{nx+n-1}-1}{q-1} \cdot \frac{q^{nx+n-2}-1}{q-1} \ldots \frac{q^{nx}-1}{q-1}\Gamma_q([nx])$, in which, for brevity, q denotes $p^{\frac{1}{n}}$, and $\frac{q^{nx+n-1}-1}{q-1} = \frac{\left[x+\frac{n-1}{n}\right]}{\left[\frac{1}{n}\right]}$ we have

$$\Gamma_{p^{\frac{1}{n}}}([nx+n]) = \left[x+\frac{n-1}{n}\right]\left[x+\frac{n-2}{n}\right]\dots[x]\times\left[\frac{1}{n}\right]^{-n}\Gamma_{p^{\frac{1}{n}}}([nx]).$$

Let $f(x)$ denote

$$\frac{\Gamma_p([x])\,\Gamma_p\left(\left[x+\frac{1}{n}\right]\right)\dots\Gamma_p\left(\left[n+\frac{n-1}{n}\right]\right)}{\Gamma_{p^{\frac{1}{n}}}([nx])}\times\frac{1}{\left[\frac{1}{n}\right]^{nx-1}};$$

then, by means of the difference equation, we obtain

$$f(x+1) = f(x).$$

If now throughout the infinite products, all expressions of the type p^x denote principal values only, then $f(x)$ is a single-valued function with a period unity, and for positive values of x has no singularities. It must therefore be a constant, and we find

$$f(x) = f\left(\frac{1}{n}\right) = \Gamma_p\left(\left[\frac{1}{n}\right]\right)\Gamma_p\left(\left[\frac{2}{n}\right]\right)\dots\Gamma_p\left(\left[\frac{n-1}{n}\right]\right).$$

So that finally

$$\frac{\Gamma_p([x)]\,\Gamma_p\left(\left[x+\frac{1}{n}\right]\right)\dots\Gamma_p\left(\left[x+\frac{n-1}{n}\right]\right)}{\left[\frac{1}{n}\right]^{nx-1}\Gamma_{p^{\frac{1}{n}}}([nx])}$$

$$= \frac{\omega^{\frac{n-1}{2}}}{\sqrt{S\left(\frac{\omega}{n}\right)S\left(\frac{2\omega}{n}\right)\dots S\left(\frac{\overline{n-1}\,\omega}{n}\right)}} \quad \dots\dots \quad (5),$$

$S(a)$ denoting a pseudo-periodic function analogous to $\sin a$. We define $S(a)$ as

$$\frac{\omega}{S(\omega a)} = \Gamma_p([a])\,\Gamma_p([1-a]), \qquad \omega^{\frac{1}{2}} = \Gamma_p([\tfrac{1}{2}]).$$

When $p = 1$, this theorem reduces to the multiplication-theorem of Gauss and Legendre.

4. In this article a generalisation of the function x^n will be formed by means of the generalised factorial Γ_p.

To deal with simple forms first. In $[n]!$ we replace p by p^2, and obtain

$$\frac{p^2-1}{p^2-1}\cdot\frac{p^4-1}{p^2-1}\dots\frac{p^{2n}-1}{p^2-1} = \Gamma_{p^2}([n+1]) \quad (n \text{ positive and integral}),$$

so that

$$(1+p)(1+p^2)\dots(1+p^n) = (p+1)^n\frac{\Gamma_{p^2}([n+1])}{\Gamma_p([n+1])}.$$

The expression $(1+p)(1+p^2)\ldots(1+p^n)$ is written $(2)_n$, this notation being both natural and convenient in investigating the properties of the generalised Bessel function*

$$\mathrm{J}_{[n]}(x,\lambda) = \Sigma(-1)^2 \frac{\lambda^{n+2r} x^{[n+2r]}}{[r]!\,[n+r]!\,(2)_r\,(2)_{n+r}}.$$

Similarly

$$(3)_n = (1+p+p^2)^n \frac{\Gamma_{p^3}([n+1])}{\Gamma_p([n+1])}.$$

When x and n are both positive integers,

$$(x)_n = (1+p+p^2+p^{x-1}) \frac{\Gamma_{p^x}([n+1])}{\Gamma_p([n+1])}.$$

The function we are seeking is clearly for all values of x and n (subject to limitations of convergence)—

$$(x)_n = [x]^n \frac{\Gamma_{p^x}([n+1])}{\Gamma_p([n+1])} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

which reduces if $p = 1$, to the function x^n.

5. *Extension of Lommel's product.*—

$$\mathrm{J}_m(x)\mathrm{J}_n(x) = \sum_{r=0}^{r=\infty}(-1)^r \frac{\binom{m+n+2r}{r}}{\Gamma(m+r+1)\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{m+n+2r},$$

To illustrate the use of the function $(2)_n$, we take

$$\mathrm{J}_{[n]}(x) = \sum_{r=0}^{r=\infty}(-1)^r \frac{x^{n+2r}}{[r]!\,[n+r]!\,(2)_r(2)_{n+r}},$$

$$\mathfrak{J}_{[m]}(x) = \sum_{r=0}^{r=\infty}(-1)^r \frac{x^{m+2r}}{[r]!\,[m+r]!\,(2)_r\,(2)_{m+r}} p^{2r(m+r)},$$

in which m and n are not restricted to integral values, so that $[n+r]!$ denotes $\Gamma_p([n+r+1])$. The function $\mathfrak{J}_{[m]}$ may be derived from $\mathrm{J}_{[m]}$ by inverting the base p, when $\mathrm{J}_{\overline{[m]}}$ becomes $p^{m^2}\mathfrak{J}_{[m]}\left(\frac{x}{p}\right)$. We proceed to show that

$$\mathrm{J}_{[n]}(x)\,\mathfrak{J}_{[m]}(x)$$
$$= \Sigma(-1)^r \frac{\{2m+2n+4r\}!}{\{2m+2n+2r\}!\,\{2m+2r\}!\,\{2n+2r\}!\,\{2r\}!} x^{m+n+2r} \ldots\ldots \quad (7),$$

$\{2s\}!$ denoting $(2)_s\Gamma_p([s+1])$, or $[2]^s\,\Gamma_{p^2}([s+1])$.

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* 'Roy. Soc. Edin. Trans.,' vol. 41, part 1, Nos. (1), (6). 'Lond. Math. Soc. Proc.,' Series 2, vols. 1 and 2.

Multiplying together the two infinite series J and $\mathfrak{J}$, the coefficient of x^{m+n+2r} in the resulting series is

$$(-1)^r \sum_{s=0}^{s=r} p^{2s(m+s)} \frac{1}{\{2m+2s\}!\,\{2n+2r-2s\}!\,\{2r-2s\}!\,\{2s\}!} \quad \cdots \quad (8).$$

We can sum this series of $r+1$ terms simply, as follows :—

In the case ($p=1$), we see that the expression is Vandermonde's, and in the general case it is included under the following extension of Vandermonde's theorem

$$[x+y]_r = [x]_r + \sum_{1}^{r} p^{s(x-r+s)} \frac{[r]!}{[r-s]!\,[s]!} [x]_{r-s} [y]_s \quad \ldots\ldots \quad (9).$$

Substituting $m+r$ for x, $n+r$ for y, and changing the base p into p^2, we find

$$\{2m+2n+4r\}_r$$
$$= \{2m+2r\}_r + \sum p^{2s(m+s)} \frac{\{2r\}!}{\{2r-2s\}!\,\{2s\}!} \{2m+2r\}_{r-s} \{2n+2r\}_s \quad (10),$$

in which, since $r-s$ is integral,

$\{2m+2r]_{r-s}$ denotes $[2m+2r]\,[2m+2r-2]\ldots$ to $r-s$ factors.

Dividing both sides of (10) by $\{2m+2r\}!\,\{2n+2r\}!$ (m and n unrestricted) we obtain

$$\frac{\{2m+2n+4r\}_r}{\{2m+2r\}!\,\{2n+2r\}!\,\{2r\}!}$$
$$= \sum_{s=0}^{s=r} p^{2s(m+s)} \frac{1}{\{2m+2s\}!\,\{2n+2r-2s\}!\,\{2r-2s\}!\,\{2s\}!} \cdots \quad (11),$$

which series we have seen to be coefficient of x^{m+n+2r}, so that (7) is established. In the notation of the generalized gamma function

$$J_{[n]}(x)\,\mathfrak{J}_{[m]}(x) = \mathfrak{J}_{[n]}(x)\,J_{[m]}(x)$$
$$= \sum_{r=0}^{r=\infty} (-1)^r \frac{\Gamma_{p^2}([m+n+2r+1])}{\Gamma_{p^2}([m+n+r+1])\Gamma_{p^2}([m+r+1])\,\Gamma_{p^2}([n+r+1])\Gamma_{p^2}([r+1])}$$
$$\left(\frac{\lambda}{[2]}\right)^{m+n+2r} \quad \ldots\ldots\ldots \quad (12).$$

By means of this product various series of squares of Bessel functions, and series of products of pairs of Bessel functions, may be generalized ; for example,

$$\frac{2x}{\pi} = \{J_{\frac{1}{2}}\}^2 + 3\,\{J_{\frac{3}{2}}\}^2 + 5\,\{J_{\frac{5}{2}}\}^2 + \ldots. \quad \text{(Lommel.)}$$

corresponds to

$$\frac{x}{[2]}\left\{\frac{1}{\Gamma\left(\left[\frac{3}{2}\right]\right)}\right\}^2 = \Sigma\, p^{r(r-1)}\frac{[4r+2]}{[2]}\,\mathrm{J}_{\left[\frac{2r+1}{2}\right]}(x)\,\mathfrak{J}_{\left[\frac{2r+1}{2}\right]}(x) \ldots \quad (13).$$

It is easily established that

$$\mathrm{J}_{[n]}\,\mathfrak{J}_{[1-n]} + \mathrm{J}_{[-n]}\,\mathfrak{J}_{[-1+n]} = \frac{[2]}{x\Gamma_{p^2}([n])\,\Gamma_{p^2}([1-n])} \quad \ldots \quad (14).$$

This reduces to

$$\mathrm{J}_n\,\mathrm{J}_{-n+1} + \mathrm{J}_{-n}\,\mathrm{J}_{n-1} = \frac{2}{\pi x}\sin n\pi.$$

As remarked in article (3) $\dfrac{1}{\Gamma_{p^2}([n])\,\Gamma_{p^2}([1-n])}$ is a pseudo-periodic function of n analogous to $\sin n\pi$. The period is given by

$$\omega^{\frac{1}{2}} = \Gamma_{p^2}\left(\left[\tfrac{1}{2}\right]\right).$$

6. *The Function* $\mathrm{B}_p([x]\,[y])$.—Let $\mathrm{F}([n-1]\,x^{p^m})$ denote the convergent infinite series

$$1 - p\frac{[n-1]}{[1]}x^{p^m} + \ldots + (-1)^r p^{r(r+1)/2}\frac{[n-1]\ldots[n-r]}{[r]\,!}x^{p^m[r]} + \ldots$$
$$\ldots\ldots\ldots\ldots \quad (15).$$

If $p = 1$, this series reduces to $(1-x)^{n-1}$.

Consider

$$p^{-m}\int_0^1 \mathrm{F}([n-1]\,x^{p^m})\,x^{p[m-1]}dx = \mathrm{B}_p([m],[n]) \quad \ldots\ldots \quad (16).$$

Integrating the series term by term, we obtain, after obvious reductions

$$\frac{p^{-m}}{[m]}\left\{1 + p^n\frac{[1-n]\,[m]}{[1]\,[m+1]} + p^{2n}\frac{[1-n]\,[2-n]\,[m]\,[m+1]}{[1]\,[2]\,[m+1]\,[m+2]} + \ldots\right\}.$$

The series within the large brackets is a particular case of Heine's series

$$1 + p^{\gamma-\alpha-\beta}\frac{[\alpha]\,[\beta]}{[1]\,[\gamma]} + \ldots = \prod_{n=0}^{\infty}\frac{[\gamma-\beta+n]\,[\gamma-\alpha+n]}{[\gamma-\alpha-\beta+n]\,[\gamma+n]}$$

$$= p^{\alpha\beta}\frac{\Gamma_p([\gamma-\alpha-\beta])\,\Gamma_p([\gamma])}{\Gamma_p([\gamma-\alpha])\,\Gamma_p([\gamma-\beta])} \quad (p>1),$$

whence

$$\frac{1}{p^{mn}}\frac{\Gamma_p([m])\,\Gamma_p([n])}{\Gamma_p([m+n])} = \mathrm{B}([m],[n]) \quad \ldots\ldots\ldots\ldots \quad (17).$$

7. *The Multiplication Theorem for the Function* B_p :—

$$\frac{\mathrm{B}_p([x],[\gamma])\ldots\mathrm{B}_p\left(\left[x+\frac{n-1}{n}\right],[y]\right)}{\mathrm{B}_{p^{\frac{1}{n}}}([y],[y])\ldots\mathrm{B}_{p^{\frac{1}{n}}}([\overline{n-1}\,.\,y],[y])}$$

$$=\mathrm{B}_{p^{\frac{1}{n}}}([nx],[ny])\times\frac{p^{(n-1)y(y-1)/2-nxy}}{\left\{\left[\frac{1}{n}\right]\left(\frac{1}{n}\right)_{y-1}\right\}^n}\;\ldots\ldots\quad(18).$$

A particular case of this is

$$\mathrm{B}_p([x],[x])\,\mathrm{B}_p([x+\tfrac{1}{2}]\,[x+\tfrac{1}{2}])=\frac{\{(\frac{1}{2})_{2x-1}\}^2\,\{\Gamma_p([\frac{1}{2}])\}^2}{[2x]}p^{-x^2-(x+\frac{1}{2})(x+\frac{1}{2})}\quad(19).$$

If $p=1$, we obtain

$$\mathrm{B}(x,x)\,\mathrm{B}(x+\tfrac{1}{2},x+\tfrac{1}{2})=\frac{\pi}{2^{4x-1}x}\qquad\text{(Binet.)}$$

8. *The Logarithmic Derivatives of the Function* Γ_p.—From the expression

$$\{\Gamma_p([x])\}^{-1}=p^{\frac{1}{2}x(x-1)}[x]\,e^{\mathrm{P}x}\prod_{s=1}^{s=\infty}\left\{\left(1+p^{-x}\frac{[x]}{[s]}\right)e^{-\frac{x}{[s]}}\right\}\qquad(p>1)$$

we obtain

$$\frac{d}{dx}\{\log\Gamma_p([x])\}=-\mathrm{P}-\tfrac{1}{2}\log p+x\log p=\frac{\log p}{(p-1)}\frac{p^x}{[x]}$$

$$+\sum_1^{\infty}\left\{\frac{1}{[s]}-\frac{\log p}{(p-1)}\frac{1}{[x+s]}\right\}\;\ldots\ldots\quad(20).$$

From this

$$-\mathrm{P}=\Gamma_p'([1])+\tfrac{1}{2}\log p+\sum_{s=1}^{\infty}\left(1-\frac{\log p}{p-1}\right)\frac{1}{[s]},\quad\ldots\ldots\quad(21),$$

which reduces when $p=1$ to $\Gamma'(1)=-\gamma$.

Similarly,

$$\left.\begin{aligned}
\frac{d^2}{dx^2}\{\log\Gamma_p([x])\}&=\log p+\lambda^2\sum_{s=0}^{=\infty}\frac{p^{s+x}}{[s+x]^2}\\
\frac{d^3}{dx^3}\{\log\Gamma_p([x])\}&=-\lambda^3\sum\frac{p^{s+x}(p^{s+x}+1)}{[s+x]^3}\\
\frac{d^4}{dx^4}\{\log\Gamma_p([x])\}&=\lambda^4\sum\frac{p^{s+x}(p^{2s+2x}+4p^{s+x}+1)}{[s+x]^4}\\
\cdots\cdots&\qquad\cdots\cdots\\
\cdots\cdots&\qquad\cdots\cdots
\end{aligned}\right\}\quad(22).$$

$$\lambda=\frac{\log p}{p-1}.$$

Certain series of interest in connection with the function Γ_p are—

$$\frac{\Gamma_p([x])\,\Gamma_p([c+1])}{\Gamma_p([x+c])} = \sum_{n=0}^{n=\infty}(-1)^n p^{\frac{n(n+1)}{2}-(n+1)c}\,\frac{[c][c-1][c-2]\ldots[c-n]}{[n]!}\,\frac{1}{[x+n]} \quad (23),$$

$$\frac{\Gamma_p([x])}{\Gamma_p([x+\frac{1}{2}])} = \frac{p^{\frac{1}{2}}}{[\frac{1}{2}]}\,\frac{1}{\Gamma_p([\frac{1}{2}])}\sum\frac{[\frac{1}{2}][\frac{3}{2}][\frac{5}{2}]\ldots[\frac{n-1}{2}]}{[n]!}\,\frac{1}{[x+n]} \quad (24),$$

$$\frac{d}{dx}\log\frac{\Gamma_p([x+a])}{\Gamma_p([x])} = a\log p + p^x[a]\left\{\frac{1}{[x][x+a]}+\frac{1}{[x+1][x+a+1]}+\ldots\right\}\frac{\log p}{(p-1)} \quad \ldots\ldots\ldots\ldots (25).$$

$$= \log p + \lambda\,\Sigma(-1)^n\frac{[a][a-1]\ldots[a-n+1]}{[x][x+1]\ldots[x+n-1]}\,c_n.$$

$$c_n = \frac{1}{[n]}\left\{[n]\,p^{nx+\frac{1}{2}n(n-1)} - \frac{[n][n-1]}{[2]!}\,p^{(n-1)x+\frac{1}{2}(n-1)(n-2)} + \ldots + (-1)^n\right\}.$$

In case $p = 1$, the series (25) is

$$\frac{a}{x} - \frac{1}{2}\,\frac{a(a-1)}{x(x+1)} + \frac{1}{3}\,\frac{a(a-1)(a-2)}{x(x+1)(x+2)} - \ldots$$

9. *The Function* $G_p([x])$.—A function

$$G(x) = \underset{n=\infty}{L}\left[(n+1)^{\frac{1}{2}(x-1)(x-2)}\{\Gamma(n+1)\}^{x-1}\prod_{\kappa=0}^{n-1}\frac{\Gamma(1+\kappa)}{\Gamma(x+\kappa)}\right] \quad (26)$$

with the properties

$$G(x+1) = \Gamma(x)\,G(x), \quad G(1) = 1,$$

is given in Whittaker's *Modern Analysis*, p. 201, and is there referred to Alexeiewsky. This function has been discussed in detail in a more general form by E. W. Barnes.* From (26) we obtain without difficulty

$$G(x+1) = \underset{\kappa=\infty}{L}\frac{\Gamma(1)\,\Gamma(2)\,\Gamma(3)\ldots\Gamma(\kappa)}{\Gamma(x+1)\,\Gamma(x+2)\,\Gamma(x+3)\ldots\Gamma(x+\kappa)}\{\Gamma(1+\kappa)\}^x(\kappa+1)^{\frac{1}{2}x(x-1)}.$$

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* 'Lond. Math. Soc. Proc.,' vol. 31, pp. 358 *et seq.*

Let us form a function

$$\begin{aligned} &\mathrm{G}_p([x+1]) \\ &= \mathop{\mathrm{L}}_{\kappa=\infty} \frac{\Gamma_p([1])\,\Gamma_p([2])\,\ldots\,\Gamma_p([\kappa])}{\Gamma_p([x+1])\,\Gamma_p([x+2])\,\ldots\,\Gamma_p([x+\kappa])} \\ &\qquad\qquad \{\Gamma_p([\kappa+1])\}^x\,[\kappa+1]^{\frac{1}{2}x(x-1)}\,p^{\frac{1}{6}x(x-1)(x-2)} \quad\ldots\ldots\ldots \quad (27). \end{aligned}$$

We notice that this function will reduce, factor by factor, to $\mathrm{G}(x)$, if we put $p = 1$.

Difference Equation.—

$$\mathrm{G}_p([x+1]) = \Gamma_p([x])\,\mathrm{G}_p([x]).$$

From the infinite product we have

$$\begin{aligned} \frac{\mathrm{G}_p([x+1])}{\mathrm{G}_p([x])} &= \Gamma_p([x]) \\ &\times \mathop{\mathrm{L}}_{\kappa=\infty} \frac{\Gamma_p([\kappa+1])}{\Gamma_p([\kappa+x])}\,[\kappa+1]^{\frac{1}{2}x(x-1)-\frac{1}{2}(x-1)(x-2)}\,p^{\frac{1}{6}x(x-1)(x-2)-\frac{1}{6}(x-1)(x-2)(x-3)} \\ &= \Gamma_p([x])\,p^{\frac{1}{2}(x-1)(x-2)} \mathop{\mathrm{L}}_{\kappa=\infty} \frac{\Gamma_p([\kappa+1])}{\Gamma_p([\kappa+x])}\,[\kappa+1]^{x-1}. \end{aligned}$$

In the case $(p>1)$ the evaluation of the limit is not difficult, for since

$$\Gamma_p([\kappa+1]) = [1]\,[2]\,\ldots\,[\kappa],$$

the expression

$$p^{\frac{1}{2}(x-1)(x-2)} \mathop{\mathrm{L}}_{\kappa=\infty} \frac{\Gamma_p([\kappa+1])}{\Gamma_p([\kappa+x])}\,[\kappa+1]^{x-1}$$

$$\begin{aligned} &= \mathrm{L}\,\frac{[\kappa+x]\,[\kappa+x+1]\,\ldots\,[\kappa+x-1+\kappa]}{[\kappa]^{x+\kappa-1}} \\ &\qquad\qquad [\kappa+1]^{x-1}\,p^{\frac{1}{2}(x^2-3x+2-\kappa^2-2\kappa x-x^2+\kappa+x)} \\ &= \mathrm{L}\left\{\frac{[\kappa+x]}{[\kappa]}\,\ldots\,\frac{[2\kappa+x-1]}{[\kappa]}\right\}\left(\frac{[\kappa+1]}{[\kappa]}\right)^{x-1} p^{\frac{1}{2}(2-2x-2\kappa x-\kappa^2+\kappa)} \\ &= \mathrm{L}\left\{p^{x+\overline{x+1}+\overline{x+2}+\,\ldots\,\overline{x+\kappa-1}} \times p^{x-1} \times p^{1-x-\kappa x-\frac{1}{2}(\kappa^2+\kappa)}\right\} = p^0 = 1, \end{aligned}$$

so that

$$\mathrm{G}_p([x+1]) = \Gamma_p([x])\,\mathrm{G}_p([x]) \quad\ldots\ldots\ldots\ldots\ldots \quad (28).$$